Wittgenstein's Tractatus logico-philosophicus, glossary of mathematical and logical symbols

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Wittgenstein's *Tractatus logico-philosophicus*

Glossary of Mathematical and Logical Symbols

This guide to the symbols used in the *Tractatus* is divided into the five sections:

1. Uses of Roman letters,
2. Uses of Greek letters,
3. Logical symbols,
4. Logical and mathematical forms, and
5. The grouping of expressions.

In the first four sections, the entry for each symbol describes its application in the *Tractatus* and, in some cases, its source and usage elsewhere. A fifth section discusses one special use of the dot that may be difficult for the reader unfamiliar with the notation of the *Principia Mathematica*.

The *Principia* is the source for almost all of the logical notation of the *Tractatus*.[1] It employs the dot and its forms (".", ":", ".:" or ":.", and "::"), in at least three separate ways:

1. As the logical connective "and" (conjunction),

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[4] For a further explanation, see Bernard Linsky, "The Notation of the Principia Mathematica," *Stanford Encyclopedia of Philosophy*, <http://plato.stanford.edu/entries/pm-notation/>.

2. As part of the notations for quantification (universal “(x).fx”) and existential “(∃x).fx”) — the dots here make explicit the application of quantification to the grouping of expressions that follow, and finally
3. As the indication of the grouping of expressions — the order in which the logical connectives combine their objects — what is indicated in computer languages by a specific priority for the application of the operations and by the use of parentheses (brackets in British usage).

While the section on logical symbols covers the first two uses of dots, the final section of the glossary explains a method for eliminating any ambiguity by using parentheses. The result is a notation that will already be familiar to many readers; and it is certainly a notation that is easier to understand without previous experience. All complex occurrences of the dot notation appearing in the *Tractatus* are shown in both the original and the simplified form.

In the following, “_” indicates the position for a simple item of some type used as an argument to the specified item; “(_,_)” indicates that two arguments are present within the parentheses forming an ordered list of arguments; “*” indicates a complex item (either several items combined by some logical connective or a function of a simple item or items).

Uses of Roman Letters

Following the *Principia*, a lower case letter a or b represents the proper name of a particular object. This contrasts with the letters x, y, etc., which represent name variables, i.e., those that take on the value of a particular proper name. It also contrasts with the letters p, q, etc., which represent particular propositions.

The letters a and b commonly appear in relations ("aRb"), in functions ("fa" or "f(a,b)"), and in the description of operations that may be applied repeatedly to an initial object ("O'a").

The capital letter A has two separate uses in the *Tractatus*, aside from the attempted notations for generality (*Allgemeinheit* in German, translated here as G — see the entry under **Gen.fx** below). In the first, it is the proper name of an undefined object. The sentence about the sign 'A' in which it appears, "'A' is the same sign as 'A'" (proposition 3.203) seems to reflect a statement about objects in the *Principia*, "when a text-book of logic asserts that 'A is A', without any indication as to what A may be, what is meant is that *any* statement of the form 'A is A' is true."[2] In the second use, the letter A indicates the proper name of a person: "A knows that . . ." (proposition 5.1362) and "A believes that . . .", "A has the thought . . .", "A says . . ." (proposition 5.542). This follows Russell's usage, as in his article "On the Nature of Truth and Falsehood" of 1910, where A and B are persons about whom a subject ("S" in Russell's later work) makes the judgment such as "A loves B". Russell writes: "Thus the judgment that two terms have a certain relation R is a relation of the mind to the two terms and the relation R with the appropriate sense . . ."[3] Note, however, that the text of Wittgenstein's *Notes on Logic*

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[2] Whitehead and Russell, *Principia Mathematica To *56*, 4.

[3] Bertrand Russell, "On the Nature of Truth and Falsehood," in *Philosophical Essays* (London: George Allen & Unwin, 1966), 158.

from 1913 uses a lowercase a to indicate the judging person, or at least the transcription and translation of his oral remarks has him doing so.[4]

A third use of the capital letter A occurs in the "Notes Dictated to G. E. Moore in Norway."[5] Here the implication is that A indicates a complex: "Take ϕa and ϕA: and ask what is meant by saying, 'There is a thing in ϕa, and a complex in ϕA'?" See the discussion of the capital letters P and Q below for another case in which the capital letter implies complexity.

c is also used as subscript. See the entry "$+_c$ (addition sign for cardinal numbers)," under "Logical and Mathematical Forms" below.

f_, f(_,_), F_, F(*), g_ are all notations for a function.

Def. is used in the notation for definition. See the entry "_=_ Def. (definition)," under "Logical and Mathematical Forms" below.

Gen.fx, f(x_g), (G,G).F(G,G) are translations of Alg.fx, f(x_a), (A,A).F(A,A), *generality* being a translation of the German *Allgemeinheit*. They are all attempts at finding a proper notation for generality, which are superceded by the truth-function of elementary propositions. See propositions 4.0411 and 5.

K_n, L_n are values, respectively, for the discrete possibilities of n binary states, and the combinations of those possibilities. See the entries for these formulas under "Logical and Mathematical Forms" below.

N($\bar{\xi}$) is a complex expression involving a special meaning for N. Here it indicates the NOR-operation applied to any arbitrary term in the series that defines the general

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[4] Wittgenstein, *Notebooks*, 95.

[5] Ibid., 111-12.

form of the elementary proposition. See chapter 3 and under “Logical and Mathematical Forms” below.

n is a numerical variable. See the entries for the formulas for the discrete possibilities of n binary states, and the combinations of those possibilities under “Logical and Mathematical Forms” below.

O’_ is the notation for an operation applied to the argument as it appears at propositions 5.2521 and 5.2522. The repetitions of the operation can be represented with a superscript just before the apostrophe. The Greek Omega is substituted in some cases; see below under the uses of Greek letters. Also see the entry “_’_”, under “Logical and Mathematical Forms” below.

p, q, r, s, P, Q, R, . . . represent propositions. The lower-case letters indicate elementary propositions. The upper-case letters appear only in proposition 5.501; apparently there they imply that the truth-table method that Wittgenstein develops as the general form of the proposition can also apply to non-elementary propositions. This overly subtle hint points to a technical innovation that first appeared in the *Tractatus*, since the truth-tables had previously only applied to elementary propositions.[6]

_R_ indicates a relation between the first argument and the second. This is one possible form for a proposition.

T, F as in (FFFT)(p,q) indicate truth-possibilities of the truth-function indicate the binary possibilities: true or false. They represent translations of the W and F used in German, indicating *wahr* and *falsch*.

T_r, T_{rs} represent, respectively, the truth-grounds of the propositions r and those that are common to r and s. See propositions 5.15 and 5.151.

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[6] Hacker, *Wittgenstein’s Place*, 280, n. 18.

x, y, u, . . . are name variables, which will take the values of the proper names of specific objects, such as a and b.

Uses of Greek Letters

η (lower-case eta), along with lower-case xi, is used as a variable for an expression where, as Anscombe notes, it is "not tied to any one kind, as is x, which is a name variable, or again n, which is a numerical variable: used in informal exposition by Frege and Wittgenstein."[7]

κ (lower-case kappa) appears as a special variable in the Sigma notation (for mathematical summation) discussed below. It represents the possible combinations of the discrete possibilities of n binary states, i.e., how many of the K_n cases are taken at a time.

μ (lower-case mu), ν (lower-case nu) are used for numerical variables. Both appear as superscripts in the notations for repeated application of an operation (propositions 6.02 and 6.241). Lower-case nu is also used in the formula for the discrete possibilities of n binary states.

ξ (lower-case xi) is used as a variable, when the kind is not specified. See lower-case eta above.

Σ (upper-case Sigma) is used in the standard mathematical notation for summation of the quantities in a given collection of values of a variable.

In the standard Sigma notation, the equation written under the Sigma indicates the index and its first value to the collected variable (e.g., $\nu=0$) and the quantity written above the Sigma indicates the last value of the index (e.g., n), meaning "for each ν from zero to n". Some confusion was created by Wittgenstein's earlier use of an variant of this notation, in which the starting value alone was written below the Sigma and the index appeared to the right of it.[8]

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[7] Anscombe, *An Introduction*, 24.

[8] Wittgenstein, *Letters to C. K. Ogden*, 48.

For the specific use, see the entries for the formulas for the discrete possibilities of n binary states and for their possible combinations, under “Logical and Mathematical Forms” below.

ϕ (lower-case Phi), ψ (lower-case Psi) are used in the notation for function, as an alternative to the Roman letters f and g; see above, under “Uses of Roman Letters.”

Ω (upper-case Omega) is used as an alternative to capital O, which indicates the variable sign for an operation (see above, under “Uses of Roman Letters”), in propositions 6.01, 6.02, and 6.241. In 6.241, the superscripts to the operation, which indicate the number of repetitions of the operation to obtain a particular result, are also indicated with Greek letters (ν and μ, see above). Anscombe suggests that the Greek letter is used in preference to the Roman since O^0 (“Oh-superscript-zero”) is “disagreeably unperspicuous.” She considers the use of Greek letters, as opposed to Roman, in the superscripts to be “pointless.”[9]

[9] Anscombe, *An Introduction*, 125, n. 1.

Logical Symbols

⊢ (assertion) is a sign used by Gottlob Frege, and in the *Principia*, indicating the proposition is asserted or judged to be true. See proposition 4.442.

_._ (conjunction) is the sign for AND when it does not serve in one of the other roles indicated above. This results in the logical product of the arguments that it connects.

_∨_ (disjunction) is the sign for OR, sometimes called non-exclusive in that the result is true when both arguments are true, as well as when either one is true alone. This results in the logical sum of the propositions it connects.

~_, ~(*) (negation) is the sign for NOT, as it applies to a single argument.

_|_ (complex negation or rejection) is the sign for NOR, the rejection of the arguments it connects, following Henry Maurice Sheffer.

_⊃_ (implication) is the sign for IMPLIES, meaning that either the first proposition is false or the second one is true. Anscombe says, “This (minimum) sense of ‘if . . . then’ occurs in ‘If that is so, I’m a Dutchman’, which if I am known not to be a Dutchman is a way of saying that ‘that’ is *not* so.”[10]

_≡_ (equivalence) is the sign for equivalence, meaning that both the first argument implies the second and the second implies the first. Either both arguments are true, or both are false.

_=_ (identity or mathematical equality) is the sign for identity, indicating that one side is the same as the other. It is also used in formulas of propositions 4.27 and 4.42 to indicate equality in quantity.

_=_ Def. (definition) is the sign for an identity that is true by definition.

[10] Ibid., 23.

(_).*, (_,_).* (universal quantification) indicates that the function or other complex argument in the second place holds for all values in the first argument. This expresses the notion ‘for all arguments, such and so holds’.

(∃_).*, (∃_,_).* (existential quantification) indicates that the function or other complex argument in the second place holds for some (at least one value) of the first argument. This expresses the notion ‘for some argument . . .’ or ‘there exists an argument such that such and so holds’.

Logical and Mathematical Forms

$\overline{p}$, $\overline{\xi}$, N($\overline{\xi}$) (p-bar, xi-bar, N of xi-bar) are the components in a series (see next entry). It is very likely that the superscript-bar in these notations actually is a vinculum, a mark of aggregation. Even though the items under the bar appear to be simple, they each represent one or more elementary propositions. Such a bar is also used in a related sense to indicate the arithmetic mean, and it appears in that way in the *Mechanics* of Hertz. As used in the *Tractatus*, $\overline{p}$ would indicate any possible quantity of elementary propositions taken as an ensemble: (p), (p,q), or (p,q,r), and so forth, where p conventionally indicates the first of the group. $\overline{\xi}$ then represents any one of the discrete possibilities produced by the combination of these elementary propositions being true or false. N($\overline{\xi}$) represents the application of the NOR-operation on the previous term, where this operation involves both combining and negating some or all of the elementary propositions. This procedure is discussed in chapter 3.

[_,_,_] is the notation for a series, following Giuseppi Peano. The first argument represents the start of the series; the second, an arbitrary term in the series; and the third, the subsequent term. The third argument thus shows the operation performed on a term in a series to advance to the next term. For example, [0,n,n+1)] represents the series of cardinal numbers.

_'_ is the notation for the application of an operation. The first term indicates the operation; the second, the term to which the operation is applied. The repetitions of the operation can be represented with a superscript just before the apostrophe.

[_,_]'_ is, as above, a way of representing an operation, here defining the operation itself, shown in square brackets, as any term in a series followed by the next term; these are the same as the second and third terms in the notation for a series (see above). The repetitions of the operation can be represented with a superscript just before the apostrophe.

Subscript or index is used in the *Tractatus* as an affix to indicate the application of an operation or term. See above for K_n, L_n, T_r, and T_{rs}. See below for $\aleph_0$ and $+_c$.

Superscript is used in the *Tractatus* as the indication of the repeated application of an operation. See propositions 6.02 and 6.241.

$\aleph_0$ (aleph-null) represents the first transfinite cardinal number, following Georg Cantor. See proposition 4.1272.

$+_c$ (addition sign for cardinal numbers) is from the *Principia* notation. See proposition 5.02, which makes the point that the subscript itself is not a separate sign.

The binomial coefficient (shown below) gives the number of combinations of n things taken ν at a time. Note here the mathematical distinction between permutations, which are ordered arrangements of elements, and combinations, which are sets of elements without regard to order. Thus, AB and BA are different permutations, but the same combination, of two letters.

For integers n, ν, where $n \geq \nu \geq 0$; and where $n!$ represents the factorial of n, i.e., the product of each of the integers between one and n, the formula for the binomial coefficient is as follows:

$$\binom{n}{\nu} = \frac{n!}{(n-\nu)!\,\nu!}$$

The values of the binomial coefficient are given in Pascal's triangle:

$\nu =$

$n =$

	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Formula for the discrete possibilities of n binary states.

$$K_n = \sum_{\nu=0}^{n} \binom{n}{\nu}$$

This is the sum of each number of combinations of n things taken ν at a time, varying ν from zero to n. See proposition 4.27.

Formula for the possible combinations of the discrete possibilities of n binary states.

$$L_n = \sum_{\kappa=0}^{K_n} \binom{K_n}{\kappa}$$

In this formula the number of cases is replaced by the number of discrete possibilities of n binary states, or K_n. And ν (lower-case nu) is replaced by κ (lower-case kappa). See proposition 4.42.

The Grouping of Expressions

The following is a method of substituting parentheses (or brackets) for the dot notation as it is used to indicate the scope of an operation on the surrounding expressions. These rules provide for the ultimate elimination of dots by replacing them with parentheses to indicate the scope of an operation, with a dotless notation for quantification, and with "&" in the case of conjunction.

For the purpose of these rules, an *expression* is defined as any part of a proposition that either appears within parentheses itself or is terminated on the right and left by an operation or by the beginning or end of the proposition. An *operation* is any of the following: the dot and its forms (".", ":", ".:" or ":.", and "::"), "∨", "⊃", "|", "≡", or "=". Simple negation, "~", is not an operation for the purposes of these rules, because it only affects the expression immediately to its right. In the *Principia Mathematica*, wherever there might be any ambiguity, complex expressions being negated are put in parentheses. Wittgenstein is not fully explicit about this, and expressions like ~a=b need to be read as ~(a=b) — i.e., the expression is equivalent to "a does-not-equal b", rather than "not-a equals b".

The occurrences of dots are divided into three categories, shown in decreasing scope:

Group I. Dots to the right or left of the following operations: "∨", "⊃", "|", "≡", or "=";

Group II. Dots used in the notation for quantification (i.e., having an apparent variable on the left, such as (_).* for universal quantification, or (∃_).* for existential quantification); and

Group III. Dots used alone to indicate conjunction (i.e., not appearing along with another operation, as in Group I, and not having an expression to the left that indicates quantification, as in Group II).

The general rule for producing a fully parenthesized expression is that you place parentheses around the parts of the expression with the lowest scope first, working upward, replacing or removing dots as you go.[11] The detailed rules are as follows:

1. Keep expressions already in parentheses as separate units, using the following rules to parse their internal structure.
2. Put parentheses around any two expressions connected by one of the following operations when they occur without dots adjoining: "∨", "⊃", "|", "≡", or "=".
3. Put parentheses around any expressions connected by a single dot of Group III (conjunction) and replace the dot with "&". These may occur in series, which can be surrounded with a single parentheses, e.g. (aRx & xRy & yRb).
4. Put parentheses around the second of any two expressions connected by a single dot of Group II (quantification) and remove the dot — the convention here is that quantification and other apparent variables can be indicated by the pair of expressions in parentheses.
5. Find any expressions to the left or right of a single dot of Group I (i.e., when the expression and one of the following operations are separated by a single dot: "∨", "⊃", "|", "≡", or "="); and then remove the single dot — it is made unnecessary by the parentheses already added.
6. Repeat rules 3, 4, and 5 for double dots, adding parentheses and removing dots.

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[11] There are other ways to do this, such as working from the largest scope to the smallest. It is also useful to consider Copi's discussion of the use of dots as brackets. He starts from a notation using a centered dot for AND along with the other symbols for operations, and he shows the scope of the operation using a hierarchy of brackets, { [( )] }. Then he reconstructs the original dot notation by replacing the brackets with dots, making adjustments for symmetry and easy of reading. See Irving M. Copi, *Symbolic Logic*, 4th ed. (New York: Macmillan and Co., 1973), 227-230.

7. Repeat rule 6 for triple dots, quadruple dots, etc., until there are only pairs of expressions for apparent variables or single expressions, with the following operations without dots connecting them: "&", "∨", "⊃", "|", "≡", or "=". All complex expressions will appear within one or more sets of parentheses.

The following is a list of all complex uses of the dot notation as they appear in the *Tractatus*. Each entry is followed by its form in a simplified notation. In this notation, the parentheses are embedded within parentheses to indicate the scope of each operation. Only single dots remain as used to indicate conjunction.

A number of these propositions are rewritten by Wittgenstein in another form or they are questioned by Wittgenstein as being all together unsayable. In these cases the first form is preceded by a question mark (?).

There are a number of minor deviations from the standard notation used in the *Principia Mathematica*, which suggests that Wittgenstein used it somewhat casually. As mentioned above, we need read expressions like ~a=b as ~(a=b). There are a few cases where Wittgenstein uses unnecessary dots, e.g. q:p∨~p, which could be written q.p∨~p without any lose. In other cases, we need to see that a more explicit form is necessary to follow the standard of the *Principia*. Both the original form and a more explicit form are shown below in these cases. One proposition occurs in two different forms in the original printing at propositions 5.5301 and 5.5321; the second occurrence is less explicit than the first one but any alternative possible interpretation is unlikely. Again in 6.1201 and 6.1221, one proposition as finally parsed appears in two forms; here unnecessary parentheses appear in the form at 6.1201 — it is certainly possible in this case that a careful editor might have done so. Finally, there are equations appearing in propositions 5.51 and 5.52 that do not follow the fully explicit use of parenthetical dots; here, however, the strict application of the rules would contradict the sense of the equations. In these cases, the equal sign without dots must be taken to have greater scope than the operation in the part of the expression that follows it.

The complete list of dotted propositions and their corrected and simplified forms follows:

3.333 $(\exists\phi):F(\phi u).\phi u=Fu$

$(\exists\phi)(F(\phi u)\ \&\ (\phi u = Fu))$

4.1252 $(\exists x):aRx.xRb$

$(\exists x)(aRx\ \&\ xRb)$

$(\exists x,y):aRx.xRy.yRb$

$(\exists x,y)(aRx\ \&\ xRy\ \&\ yRb)$

4.1273 Same as in 4.1252

5.101 $p\supset p.q\supset q$

$(p \supset p)\ \&\ (q \supset q)$

$p.\sim q{:}\vee{:}q.\sim p$

$(p\ \&\ \sim q) \vee (q\ \&\ \sim p)$

5.1311 $p|q.|.p|q$

$(p \mid q) \mid (p \mid q)$

5.441 $(\exists x).fx.x=a$

$(\exists x)(fx\ \&\ (x = a))$

5.47 Same as in 5.441

5.51	$N(\bar{\xi}) = \sim p.\sim q$	Original form
	$N(\bar{\xi}).=.\sim p.\sim q$	More explicit form
	$N(\bar{\xi}) = (\sim p \;\&\; \sim q)$	
5.513	$q{:}p \vee \sim p$	
	$q \;\&\; (p \vee \sim p)$	
5.52	$N(\bar{\xi}) = \sim(\exists x).fx$	Original form
	$N(\bar{\xi}).=.\sim(\exists x).fx$	More explicit form
	$N(\bar{\xi}) = (\sim(\exists x)fx)$	
5.5301	? $(x){:}fx.\supset.x=a$	See variant in 5.5321
	$(x)(fx \supset (x = a))$	
5.531	? $f(a,b).a=b$	
	$f(a,b) \;\&\; (a = b)$	
	? $f(a,b).\sim a=b$	Read $\sim a=b$ as $\sim(a=b)$
	$f(a,b) \;\&\; \sim(a = b)$	
5.532	? $(\exists x,y).f(x,y).x=y$	
	$(\exists x,y)(f(x,y) \;\&\; (x = y))$	
	? $(\exists x,y).f(x,y).\sim x=y$	Read $\sim x=y$ as $\sim(x=y)$
	$(\exists x,y)(f(x,y) \;\&\; \sim(x = y))$	
	$(\exists x,y).f(x,y).\vee.(\exists x).f(x,x)$	
	$(\exists x,y)(f(x,y)) \vee (\exists x)(f(x,x))$	

5.5321 ? (x):fx⊃x=a — Original form

(x):fx.⊃.x=a — More explicit form (see also 5.5301)

(x)(fx ⊃ (x = a))

(∃x,y).fx.⊃.fa:~(∃x,y).fx.fy

(∃x,y)(fx ⊃ fa) & ~(∃x,y)(fx & fy)

(∃x).fx:~(∃x,y).fx.fy

(∃x)(fx) & ~(∃x,y)(fx & fy)

5.534 ? a=b.b=c.⊃.a=c

((a = b) & (b = c)) ⊃ (a = c)

? (x).x=x

(x)(x = x)

? (∃x).x=a

(∃x)(x = a)

5.5352 ? ~(∃x).x=x

~(∃x)(x = x)

6.1201 (p⊃q).(p):⊃:(q) — Original form (see also 6.1221)

(p⊃q).p:⊃:q — Dropping unnecessary parentheses

((p ⊃ q) & p) ⊃ q

(x).fx:⊃:fa

(x)(fx) ⊃ fa

6.1221 p⊃q.p

(p ⊃ q) & p

p⊃q.p:⊃:q See variant in 6.1201

((p ⊃ q) & p) ⊃ q